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Stable isomorphism of dual operator spaces [☆]

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Abstract

We prove that two dual operator spaces X and Y are stably isomorphic if and only if there exist completely isometric normal representations ϕ and ψ of X and Y , respectively, and ternary rings of operators M_1, M_2 such that $\phi(X) = [M_2^* \psi(Y) M_1]^{-w*}$ and $\psi(Y) = [M_2 \phi(X) M_1^*]^{-w*}$. We prove that this is equivalent to certain canonical dual operator algebras associated with the operator spaces being stably isomorphic. We apply these operator space results to prove that certain dual operator algebras are stably isomorphic if and only if they are isomorphic. Consequently, we obtain that certain complex domains are biholomorphically equivalent if and only if their algebras of bounded analytic functions are Morita equivalent in our sense. Finally, we provide examples motivated by the theory of CSL algebras.

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1. Introduction

K. Morita [15] developed an equivalence for rings based on their categories of modules and proved three central theorems explaining this equivalence relation. A parallel Morita theory for C^* - and W^* -algebras was introduced by Rieffel in [18]. Later Brown, Green and Rieffel [7]

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introduced the idea of stable isomorphism and proved that two C^* -algebras with strictly positive elements are strongly Morita equivalent if and only if they are stably isomorphic in the sense that the two C^* -algebras obtained by tensoring with the C^* -algebra of all compact operators on a separable Hilbert space are $*$ -isomorphic. This type of stable isomorphism theorem is often referred to as the fourth Morita theorem, and can often be used as an efficient way to prove some of the first three Morita theorems. After the advent of the theory of operator spaces and operator algebras, a parallel Morita theory for non-selfadjoint operator algebras was developed by Blecher, Muhly and the second named author in [4]. Many of the technical results needed to extend this theory to the setting of dual operator algebras appear in the book of Blecher and Le Merdy [3]. In [11] the first named author developed a version of Morita theory for dual operator algebras using a relation called Δ -equivalence, together with a certain category of modules over the algebras, and analogues of the first three Morita theorems were proved. In [13] the first and second named authors developed the fourth part of the Morita theory, stable isomorphism, for Δ -equivalence. A different Morita theory for dual operator algebras has been formulated and studied by Blecher and Kashyap [2,14]. They have shown that their equivalence relation is a coarser equivalence relation than Δ -equivalence, and have successfully proved the first three Morita theorems in their theory. In [12] the first author proved that the equivalence relation of Blecher and Kashyap is strictly coarser and that consequently, the usual stable isomorphism theorem cannot hold in their setting. We conjecture that their equivalence relation is equivalent to $(1 + \epsilon)$ -stable isomorphism.

In this paper we extend the results of [11] and [13] to dual operator spaces. We define Δ -equivalence for dual operator spaces and show that two dual operator spaces are stably isomorphic if and only if they are Δ -equivalent. Thus, we are able to develop parts of the Morita theory in a setting where the basic objects of study are not even rings. This result and several of its corollaries are included in Section 2. We end this section by applying our results for spaces to obtain some new results about algebras. In Section 3 we provide examples arising from the theory of CSL algebras.

Our notation is standard. If H and K are Hilbert spaces we denote by $H \otimes K$ their Hilbert space tensor product. For a subset $S \subseteq B(H, K)$ we denote by S' the commutant of S , by $[S]$ the linear span of S and by $\overline{[S]}^{w^*}$ the w^* -closed hull of $[S]$. If $H' \subseteq H$ is a closed subspace we let $P_{H'}$ be the orthogonal projection from H onto H' . By $\text{Ball}(X)$ we denote the unit ball of a Banach space X . For an operator algebra A we denote by $\text{pr}(A)$ the set of all projections in A .

Throughout the paper, we use extensively the basics of Operator Space Theory and we refer the reader to the monographs [3,9,16,17] for further details.

2. Stably isomorphic dual operator spaces

Let X be a dual operator space. A **normal representation** of X is a completely contractive w^* -continuous map $\phi: X \rightarrow B(K, H)$ where K and H are Hilbert spaces. A normal representation $\phi: X \rightarrow B(K, H)$ is called **non-degenerate** if $\overline{\phi(X)K} = H$ and $\phi(X)^*H = K$ and **degenerate**, otherwise. Note that if ϕ is a degenerate normal representation and if we set $H' = \overline{\phi(X)K}$, $K' = \overline{\phi(X)^*H}$ and define $\phi': X \rightarrow B(K', H')$ by $\phi'(x) = P_{H'}\phi(x)|_{K'}$, then ϕ' is a non-degenerate normal representation, which we shall refer to as the non-degenerate representation **obtained from** ϕ . If ϕ is completely isometric then ϕ' is completely isometric as well. If A is a unital dual operator algebra, a **normal representation** of A is a unital completely contractive w^* -continuous homomorphism $\alpha: A \rightarrow B(H)$ for some Hilbert space H .

If A and B are unital operator algebras and X is an operator space, X is called an **operator $A - B$ -module** if there exist completely contractive bilinear maps (in the sense of Christensen–Sinclair) $A \times X \rightarrow X$ and $X \times B \rightarrow X$. Recall that a bilinear map is completely contractive in the sense of Christensen–Sinclair if and only if the induced linear map is completely contractive when the domain is endowed with the Haagerup tensor norm. In this case there exist Hilbert spaces H, K , completely contractive unital homomorphisms $\pi : A \rightarrow B(H)$, $\sigma : B \rightarrow B(K)$ and a complete isometry $\phi : X \rightarrow B(K, H)$ such that $\phi(axb) = \pi(a)\phi(x)\sigma(b)$ for all $a \in A$, $x \in X$, $b \in B$ [16, Corollary 16.10]. The triple (π, ϕ, σ) is called a **CES representation** of the operator $A - B$ -module X . Moreover, replacing the original π and σ by their direct sums with completely isometric representations, if necessary, one may assume that π and σ are completely isometric. In this case the triple (π, ϕ, σ) is called a **faithful CES representation**. We recall that one surprising consequence of the existence of the CES representation is that an operator $A - B$ -module, is automatically an $A - B$ -bimodule, that is, the associativity condition, $(ax)b = a(xb)$, follows from the other assumptions.

If X and Y are dual operator spaces, we call a mapping $\phi : X \rightarrow Y$ a **dual operator space isomorphism** if it is a surjective complete isometry which is also a w^* -homeomorphism. If there exists such a mapping, we say that X and Y are **isomorphic dual operator spaces**. Similarly, if A and B are dual operator algebras, we call a mapping $\phi : A \rightarrow B$ a **dual operator algebra isomorphism** if it is a surjective complete isometry which is also a homomorphism and a w^* -homeomorphism. If there exists such a mapping, we say that A and B are **isomorphic dual operator algebras**.

In the case that A and B are unital dual operator algebras and X is a dual operator space, X is called a **dual operator $A - B$ -module** if it is an operator $A - B$ -module and the module actions are separately w^* -continuous. In this case the triple (π, ϕ, σ) can be chosen with the property that π , ϕ and σ be w^* -continuous completely isometric maps [3, Theorem 3.8.3]. We call such a triple a **faithful normal CES representation**.

Note that since X is an $A - B$ -bimodule the set $\mathcal{C} = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ is naturally endowed with a product making it into an algebra and every CES representation (π, ϕ, σ) as above yields a representation $\rho : \mathcal{C} \rightarrow B(H \oplus K)$ defined by $\rho\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} \pi(a) & \phi(x) \\ 0 & \sigma(b) \end{pmatrix}$. When (π, ϕ, σ) is a faithful CES representation, then the representation ρ endows \mathcal{C} with the structure of an operator algebra. In the case A and B are unital C^* -algebras, X is an operator $A - B$ -module and (π, ϕ, σ) is a faithful CES representation, this induced operator algebra structure on \mathcal{C} is unique; that is, any two faithful CES representations give rise to the same matrix norm structures. This fact was first pointed out in [5, p. 11] and follows from the uniqueness of the operator system structure on $\mathcal{C} + \mathcal{C}^*$ as can be seen from [20] (see also [3, 3.6.1]).

In case A and B are W^* -algebras the image of the faithful normal CES representation is w^* -closed and \mathcal{C} can be equipped with a dual operator algebra structure. We isolate the following useful consequence of the above remarks.

Proposition 2.1. *Let A_1, A_2, B_1, B_2 be W^* -algebras and X_1 (resp. X_2) be a dual $A_1 - B_1$ - (resp. $A_2 - B_2$ -) module. Let $\pi : A_1 \rightarrow A_2$, $\sigma : B_1 \rightarrow B_2$ be normal $*$ -isomorphisms and $\phi : X_1 \rightarrow X_2$ be a dual operator space isomorphism which is a bimodule map in the sense that*

$$\phi(lxr) = \pi(l)\phi(x)\sigma(r), \quad l \in A_1, x \in X_1, r \in B_1.$$

Then the map

$$\Phi: \begin{pmatrix} A_1 & X_1 \\ 0 & B_1 \end{pmatrix} \rightarrow \begin{pmatrix} A_2 & X_2 \\ 0 & B_2 \end{pmatrix}: \begin{pmatrix} l & x \\ 0 & r \end{pmatrix} \rightarrow \begin{pmatrix} \pi(l) & \phi(x) \\ 0 & \sigma(r) \end{pmatrix}$$

is a dual operator algebra isomorphism.

We recall some definitions from [11] and [13]. Let I be a set and ℓ_I^2 be the Hilbert space of all square summable families indexed by I . Recall that if H is a Hilbert space we may identify $B(\ell_I^2 \otimes H)$ with the space $M_I(B(H))$ of all matrices of size $|I| \times |I|$ with entries from $B(H)$ which define bounded operators on $\ell_I^2 \otimes H$. If $X \subseteq B(H)$ is an operator space we let $M_I(X) \subseteq M_I(B(H))$ denote the space of those operators whose matrices have entries from X . This space is denoted $M_I^w(X)$ in [6]. We define similarly $M_{I,J}(X)$ where I and J are (perhaps different) index sets. In particular, the column (resp. row) operator space $C_I(X)$ (resp. $R_I(X)$) over X is defined as $M_{I,1}(X)$ (resp. $M_{1,I}(X)$).

If $X \subseteq B(H)$ is a w^* -closed subspace, then it is easy to see that $M_I(X)$ is a w^* -closed subspace of $M_I(B(H))$. Moreover, if X is a w^* -closed subalgebra of $B(H)$, then $M_I(X)$ is a w^* -closed subalgebra of $M_I(B(H))$.

Definition 2.1. (i) [11] Let H and K be Hilbert spaces. Two w^* -closed subalgebras $A \subseteq B(H)$ and $B \subseteq B(K)$ are called **TRO-equivalent** if there exists a ternary ring of operators (TRO) $M \subseteq B(H, K)$, i.e., a subspace satisfying $MM^*M \subseteq M$, such that $A = \overline{[M^*BM]}^{w^*}$ and $B = \overline{[MAM^*]}^{w^*}$.

(ii) [11] Two dual operator algebras A and B are called **Δ -equivalent** if they possess completely isometric normal representations whose images are TRO-equivalent.

(iii) [13] Two dual operator algebras A and B are called **stably isomorphic** (as algebras), if there exists a cardinal I such that the algebras $M_I(A)$ and $M_I(B)$ are isomorphic as dual operator algebras.

It is clear that stable isomorphism is an equivalence relation and it is easy to see that the same holds for TRO-equivalence. While it is obvious that the relation of Δ -equivalence is reflexive and symmetric, it is not apparent that it is transitive. Nonetheless, the results of [11] show that it is equivalent to a certain category equivalence and hence it is also an equivalence relation. The results of [11] and [13] show that the relations of Δ -equivalence and stable isomorphism coincide.

In this paper we generalize this result to the case of dual operator spaces. We begin with the relevant definitions.

Definition 2.2. (i) Let $X \subseteq B(K_1, K_2)$ and $Y \subseteq B(H_1, H_2)$ be w^* -closed operator spaces. We say that X is **TRO-equivalent** to Y if there exist TRO's $M_1 \subseteq B(H_1, K_1)$ and $M_2 \subseteq B(H_2, K_2)$ such that $X = \overline{[M_2YM_1^*]}^{w^*}$ and $Y = \overline{[M_2^*XM_1]}^{w^*}$.

(ii) Let X and Y be dual operator spaces. We say that X is **Δ -equivalent** to Y if there exist completely isometric normal representations ϕ and ψ of X and Y , respectively, such that $\phi(X)$ is TRO-equivalent to $\psi(Y)$.

(iii) Let X and Y be dual operator spaces. We say that X and Y are **stably isomorphic** if there exists a cardinal J and a w^* -continuous, completely isometric map from $M_J(X)$ onto $M_J(Y)$, i.e., if they are isomorphic as dual operator spaces.

Blecher and Zarikian [6, Section 6.2] define two dual operator spaces X and Y to be **weakly Morita equivalent** if $M_{I_1, J_1}(X)$ and $M_{I_2, J_2}(Y)$ are completely isometrically isomorphic as dual operator spaces. Note that if $M_{I_1, J_1}(X)$ is completely isometrically isomorphic to $M_{I_2, J_2}(Y)$ for some cardinals I_1, I_2, J_1, J_2 , then for a large enough cardinal J the spaces $M_J(X)$ and $M_J(Y)$ are completely isometrically isomorphic. Thus, their definition of weakly Morita equivalent is the same as our stable isomorphism. Since one goal of our research is to prove that stable isomorphism is equivalent to a type of Morita equivalence, we believe that our terminology is clearer in our context.

It is obvious that the relation of TRO-equivalence of w^* -closed operator subspaces is reflexive and symmetric. We shall now prove that it is in fact an equivalence relation. First we note that the spaces involved can always be assumed to act non-degenerately.

Proposition 2.2. *Let X and Y be dual operator spaces, $\phi: X \rightarrow B(K_1, K_2)$, and $\psi: Y \rightarrow B(H_1, H_2)$ be completely isometric normal representations with TRO-equivalent images. If $\phi': X \rightarrow B(K'_1, K'_2)$, and $\psi': Y \rightarrow B(H'_1, H'_2)$ are the non-degenerate completely isometric normal representations obtained from ϕ and ψ , respectively, then the images of ϕ' and ψ' are TRO-equivalent.*

Proof. Recall that to make ϕ and ψ non-degenerate, we restrict to subspaces, $K'_2 = [\phi(X)K_1]$, $K'_1 = [\phi(X)^*K_2]$, $H'_2 = [\psi(Y)H_1]$, $H'_1 = [\psi(Y)^*H_2]$, where $[\mathcal{E}]$ denotes the closed linear subspace spanned by \mathcal{E} . Note that $K'_1 = [\phi(X)^*K_2] = [M_1\psi(Y)^*M_2^*K_2] \subseteq [M_1\psi(Y)^*H_2] \subseteq [M_1H'_1]$. We also have that $[M_1H'_1] = [M_1\psi(Y)^*H_2] = [M_1M_1^*\phi(X)^*M_2H_2]$. Since $M_1M_1^*\phi(X)^* = M_1M_1^*M_1\psi(Y)^*M_2^* \subseteq M_1\psi(Y)^*M_2^* \subseteq \phi(X)^*$, we have that $[M_1H'_1] \subseteq [\phi(X)^*K_2] = K'_1$. Thus, it follows that $K'_1 = [M_1H'_1]$ and it can be easily checked that $M'_1 = P_{K'_1}M_1|_{H'_1}$ is a TRO.

Similarly, one shows that $M'_2 = P_{K'_2}M_2|_{H'_2}$ is a TRO. It is now easy to verify that M'_1 and M'_2 implement a TRO-equivalence of $\phi'(X)$ and $\psi'(Y)$. \square

Proposition 2.3. *TRO-equivalence of w^* -closed operator spaces is an equivalence relation.*

Proof. We need to prove that TRO-equivalence is a transitive relation. Assume that $X \subseteq B(K_1, K_2)$, $Y \subseteq B(H_1, H_2)$ and $Z \subseteq B(R_1, R_2)$ are w^* -closed subspaces such that X is TRO-equivalent to Y and Y is TRO-equivalent to Z . By Proposition 2.2, we may assume that (the identity representations of) X, Y and Z are non-degenerate. We fix TRO's

$$M_1 \subseteq B(H_1, K_1), \quad M_2 \subseteq B(H_2, K_2), \quad N_1 \subseteq B(H_1, R_1) \quad \text{and} \quad N_2 \subseteq B(H_2, R_2)$$

such that

$$X = \overline{[M_2YM_1^*]}^{w^*}, \quad Y = \overline{[M_2^*XM_1]}^{w^*}, \quad Y = \overline{[N_2^*ZN_1]}^{w^*} \quad \text{and} \quad Z = \overline{[N_2YN_1^*]}^{w^*}.$$

By [10, Theorem 3.2], there exist $*$ -isomorphisms

$$\phi: (M_2^*M_2)' \rightarrow (M_2M_2^*)' \quad \text{and} \quad \chi: (N_2^*N_2)' \rightarrow (N_2N_2^*)'$$

such that

$$M_2 = \{T \in B(H_2, K_2): TP = \phi(P)T, \text{ for each } P \in pr((M_2^*M_2)')\}$$

and

$$N_2 = \{T \in B(H_2, R_2): TP = \chi(P)T, \text{ for each } P \in pr((N_2^* N_2)')\}.$$

Let $\mathcal{S} = pr((M_2^* M_2)' \cap (N_2^* N_2)'),$

$$\widetilde{M}_2 = \{T: TP = \phi(P)T, \text{ for each } P \in \mathcal{S}\}$$

and

$$\widetilde{N}_2 = \{T: TP = \chi(P)T, \text{ for each } P \in \mathcal{S}\}.$$

Observe that \widetilde{M}_2 and \widetilde{N}_2 are TRO's containing M_2 and N_2 , respectively. From [10, Lemma 2.2] it follows that

$$\overline{[\widetilde{M}_2^* \widetilde{M}_2]}^{w*} = \mathcal{S}' = \overline{[\widetilde{N}_2^* \widetilde{N}_2]}^{w*}.$$

We let $L_2 = \overline{[\widetilde{N}_2 \widetilde{M}_2^*]}^{w*} \subseteq B(K_2, R_2)$. The space L_2 is a TRO since

$$\widetilde{N}_2 \widetilde{M}_2^* \widetilde{M}_2 \widetilde{N}_2^* \widetilde{N}_2 \widetilde{M}_2^* \subseteq \widetilde{N}_2 \mathcal{S}' \mathcal{S} \widetilde{M}_2^* \subseteq \widetilde{N}_2 \widetilde{M}_2^* \subseteq L_2.$$

Similarly, if $\mathcal{T} = pr((M_1^* M_1)' \cap (N_1^* N_1)'),$ then there exist TRO's $\widetilde{M}_1 \supseteq M_1, \widetilde{N}_1 \supseteq N_1$ such that $\overline{[\widetilde{M}_1^* \widetilde{M}_1]}^{w*} = \mathcal{T}' = \overline{[\widetilde{N}_1^* \widetilde{N}_1]}^{w*}.$ As above, the space $L_1 = \overline{[\widetilde{N}_1 \widetilde{M}_1^*]}^{w*}$ is a TRO. Since $\mathcal{S}' Y \mathcal{T}' \subseteq Y$ we have

$$\begin{aligned} \widetilde{M}_2^* \widetilde{M}_2 Y \widetilde{M}_1^* \widetilde{M}_1 &\subseteq Y \Rightarrow M_2^* \widetilde{M}_2 Y \widetilde{M}_1^* M_1 \subseteq Y \\ &\Rightarrow M_2 M_2^* \widetilde{M}_2 Y \widetilde{M}_1^* M_1 M_1^* \subseteq M_2 Y M_1^* \subseteq X. \end{aligned}$$

Since $I_{K_2} \in \overline{[M_2 M_2^*]}^{w*}$ and $I_{K_1} \in \overline{[M_1 M_1^*]}^{w*}$ we have $\widetilde{M}_2 Y \widetilde{M}_1^* \subseteq X$ and hence $X = \overline{[\widetilde{M}_2 Y \widetilde{M}_1^*]}^{w*}.$

Similarly, we can show that

$$Y = \overline{[\widetilde{M}_2^* X \widetilde{M}_1]}^{w*}, \quad Z = \overline{[\widetilde{N}_2 Y \widetilde{N}_1^*]}^{w*} \quad \text{and} \quad Y = \overline{[\widetilde{N}_2^* Z \widetilde{N}_1]}^{w*}.$$

Now, writing ABC for $\overline{[ABC]}^{w*}$ and AB for $\overline{[AB]}^{w*}$ we have

$$L_2 X L_1^* = \widetilde{N}_2 \widetilde{M}_2^* X \widetilde{M}_1 \widetilde{N}_1^* = \widetilde{N}_2 Y \widetilde{N}_1^* = Z$$

and

$$L_2^* Z L_1 = \widetilde{M}_2 \widetilde{N}_2^* Z \widetilde{N}_1 \widetilde{M}_1^* = \widetilde{M}_2 Y \widetilde{M}_1^* = X. \quad \square$$

We will show later that Δ -equivalence of dual operator spaces is an equivalence relation. Note that if A and B are dual operator algebras, then they could be stably isomorphic as algebras

(which requires that the map implementing the stable isomorphism be an algebra homomorphism) or simply stably isomorphic as dual operator spaces. However, by the operator algebra generalization of the Banach–Stone theorem [3, Theorem 4.5.13] these two conditions are equivalent.

We recall the following main result from [13]:

Theorem 2.4. *Two dual operator algebras are Δ -equivalent if and only if they are stably isomorphic as algebras.*

In this section we shall generalize this result to the case of dual operator space. Namely, we will prove the following:

Theorem 2.5. *Two dual operator spaces are Δ -equivalent if and only if they are stably isomorphic.*

We now present the proof of one of the directions of Theorem 2.5 showing that Δ -equivalence of dual operator spaces implies stable isomorphism.

Assume, without loss of generality, that $X \subseteq B(H_1, H_2)$ and $Y \subseteq B(K_1, K_2)$ are concrete w^* -closed operator spaces which are TRO-equivalent and non-degenerate. Let $M_1 \subseteq B(H_1, K_1)$ and $M_2 \subseteq B(H_2, K_2)$ be w^* -closed TRO's such that $\overline{M_2 X M_1^*}^{w^*} = Y$ and $\overline{M_2^* Y M_1}^{w^*} = X$.

Let

$$A = \begin{pmatrix} \overline{M_2^* M_2}^{w^*} & X \\ 0 & \overline{M_1^* M_1}^{w^*} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \overline{M_2 M_2^*}^{w^*} & Y \\ 0 & \overline{M_1 M_1^*}^{w^*} \end{pmatrix}.$$

Since

$$(M_2^* M_2) X (M_1^* M_1) \subseteq M_2^* Y M_1 \subseteq X,$$

the space X is an $\overline{M_2^* M_2}^{w^*} - \overline{M_1^* M_1}^{w^*}$ -module and hence A is a subalgebra of $B(H_2 \oplus H_1)$. Since Y (resp. X) is non-degenerate, the relation $\overline{M_2 X M_1^*}^{w^*} = Y$ (resp. $\overline{M_2^* Y M_1}^{w^*} = X$) implies that $\overline{M_2 H_2} = K_2$ (resp. $\overline{M_2^* K_2} = H_2$). Thus, M_2 is non-degenerate. Taking adjoints we obtain the relations $\overline{M_1 X^* M_2^*}^{w^*} = Y^*$ and $\overline{M_1^* Y^* M_2}^{w^*} = X^*$ which imply that M_1 is non-degenerate. It follows that the (selfadjoint) algebras $\overline{M_2^* M_2}^{w^*}$ and $\overline{M_1^* M_1}^{w^*}$ are unital, and so A is unital. One sees similarly that B is a unital w^* -closed subalgebra of $B(K_2 \oplus K_1)$.

Let

$$M = \begin{pmatrix} M_2 & 0 \\ 0 & M_1 \end{pmatrix} \subseteq B(H_2 \oplus H_1, K_2 \oplus K_1).$$

Then M is a w^* -closed TRO and it is easily verified that

$$\overline{M A M^*}^{w^*} = B \quad \text{and} \quad \overline{M^* B M}^{w^*} = A.$$

By Theorem 2.4, A and B are stably isomorphic. Thus, there exists a cardinal I and a dual operator algebra isomorphism $\Phi : M_I(A) \rightarrow M_I(B)$. We have that

$$M_I(A) \simeq \begin{pmatrix} M_I(\overline{[M_2^* M_2]}^{w^*}) & M_I(X) \\ 0 & M_I(\overline{[M_1^* M_1]}^{w^*}) \end{pmatrix}$$

and

$$M_I(B) \simeq \begin{pmatrix} M_I(\overline{[M_2^* M_2]}^{w^*}) & M_I(Y) \\ 0 & M_I(\overline{[M_1^* M_1]}^{w^*}) \end{pmatrix}.$$

It is well known that Φ must carry the diagonal of $M_I(A)$ onto the diagonal of $M_I(B)$ (see e.g. [3, 2.1.2]). We claim that $\Phi\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $\Phi\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. To show this, note that $\Phi\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right)$ is a projection in the diagonal of $M_I(B)$ and hence there exist projections Q and P acting on K_2 and K_1 , respectively, such that $\Phi\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix}$. Then $\Phi\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right) = \begin{pmatrix} I-Q & 0 \\ 0 & I-P \end{pmatrix}$. Let $x \in M_I(X)$. Then

$$\begin{aligned} \Phi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) &= \Phi\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right) \\ &= \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix} \Phi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} I-Q & 0 \\ 0 & I-P \end{pmatrix} \\ &\subseteq \begin{pmatrix} QM_I(B(K_2))(I-Q) & QM_I(Y)(I-P) \\ 0 & PM_I(B(K_1))(I-P) \end{pmatrix}. \end{aligned}$$

Since Φ is surjective and Y is non-degenerate, it follows that $Q = I$ and $P = 0$. The claim is proved. Since Φ is a homomorphism, we have that $\Phi\left(\begin{pmatrix} 0 & M_I(X) \\ 0 & 0 \end{pmatrix}\right) \subseteq \begin{pmatrix} 0 & M_I(Y) \\ 0 & 0 \end{pmatrix}$ and since Φ is onto, the last inclusion is actually an equality. It follows that there exists a normal complete isometry between $M_I(X)$ and $M_I(Y)$.

In order to prove the converse direction of Theorem 2.5 we need the notion of multipliers of an operator space [3, 16]. Let X be an operator space and $M_l(X)$ be the space of all completely bounded linear maps u on X for which there exist Hilbert spaces H and K , a complete isometry $\iota: X \rightarrow B(H, K)$ and an operator $T \in B(K)$ such that $T\iota(X) \subseteq \iota(X)$ and $u(x) = \iota^{-1}(T\iota(x))$, $x \in X$. Then $M_l(X)$ can be endowed with an operator algebra structure in a canonical way and is called the **left multiplier algebra** of X . Similarly one defines the right multiplier algebra $M_r(X)$ of X . The operator space X is an operator $M_l(X) - M_r(X)$ -module; for $l \in M_l(X)$, $r \in M_r(X)$ and $x \in X$ we write $lx = l(x)$ and $xr = r(x)$. If X is a dual operator space then $M_l(X)$ and $M_r(X)$ are dual operator algebras [3, Theorem 4.7.4]. Their diagonals $A_l(X) = M_l(X) \cap M_l(X)^*$ and $A_r(X) = M_r(X) \cap M_r(X)^*$ are thus W^* -algebras. Since the maps

$$A_l(X) \times X \rightarrow X: (l, x) \rightarrow lx, \quad X \times A_r(X) \rightarrow X: (x, r) \rightarrow xr$$

are completely contractive and separately w^* -continuous bilinear maps [3, Lemma 4.7.5], the space

$$\Omega(X) = \begin{pmatrix} A_l(X) & X \\ 0 & A_r(X) \end{pmatrix} \quad (2.1)$$

can be canonically endowed with the structure of a dual operator algebra (see Proposition 2.1).

Proposition 2.6. *Let X and Y be isomorphic dual operator spaces. Then the algebras $\Omega(X)$ and $\Omega(Y)$ are isomorphic dual operator algebras.*

Proof. Assume that $\phi: X \rightarrow Y$ is a dual operator space isomorphism. We let $\sigma: M_I(X) \rightarrow M_I(Y)$ be given by $\sigma(u) = \phi \circ u \circ \phi^{-1}$. Then σ is a completely isometric homomorphism [3, Proposition 4.5.12] and we can easily check that it is w^* -continuous. Also, $\sigma(A_I(X)) = A_I(Y)$ and

$$\phi(ux) = \phi(u(x)) = \phi \circ u \circ \phi^{-1}(\phi(x)) = \sigma(u)(\phi(x)) = \sigma(u)\phi(x)$$

for all $u \in A_I(X)$, $x \in X$. Similarly, the completely isometric surjection $\tau: M_r(X) \rightarrow M_r(Y)$ given by $\tau(w) = \phi \circ w \circ \phi^{-1}$ satisfies the identity $\phi(xw) = \phi(x)\tau(w)$. The conclusion now follows from Proposition 2.1. \square

In order to complete the proof of Theorem 2.5 we will also need the following lemma.

Lemma 2.7. *Let $C_X = \begin{pmatrix} B_X & X \\ 0 & A_X \end{pmatrix}$ and $C_Y = \begin{pmatrix} B_Y & Y \\ 0 & A_Y \end{pmatrix}$ be concrete operator algebras acting on the Hilbert spaces $H_2 \oplus H_1$ and $K_2 \oplus K_1$, respectively. Suppose that B_X , A_X , B_Y and A_Y are von Neumann algebras.*

- (i) *If C_X and C_Y are TRO-equivalent, then there exist TRO's $M_1 \subseteq B(H_1, K_1)$ and $M_2 \subseteq B(H_2, K_2)$ such that $Y = \overline{M_2 X M_1^*}^{w^*}$ and $X = \overline{M_2^* Y M_1}^{w^*}$, $M_1 A_X M_1^* \subseteq A_Y$, $M_1^* A_Y M_1 \subseteq A_X$, $M_2 B_X M_2^* \subseteq B_Y$, $M_2^* B_Y M_2 \subseteq B_X$.*
- (ii) *If C_X and C_Y are Δ -equivalent, then X and Y are Δ -equivalent.*

Proof. Suppose that (i) holds and assume that C_X and C_Y are Δ -equivalent. Then there exist normal completely isometric algebra homomorphisms, $\alpha: C_X \rightarrow B(\hat{H})$ and $\beta: C_Y \rightarrow B(\hat{K})$ such that $\alpha(C_X)$ and $\beta(C_Y)$ are TRO-equivalent. Note that $\alpha(C_X)$ (resp. $\beta(C_Y)$) has the form $\begin{pmatrix} B_Z & Z \\ 0 & A_Z \end{pmatrix}$ (resp. $\begin{pmatrix} B_T & T \\ 0 & A_T \end{pmatrix}$) for a suitable decomposition $\hat{H} = \hat{H}_2 \oplus \hat{H}_1$, $\hat{K} = \hat{K}_2 \oplus \hat{K}_1$, von Neumann algebras B_Z , A_Z , B_T , A_T and w^* -closed subspaces Z , T that are isomorphic to X and Y , respectively, as dual operator spaces. Thus, (ii) follows from (i).

We now prove (i). Let P_X (resp. P_Y) denote the projection from $H_2 \oplus H_1$ onto H_1 (resp. from $K_2 \oplus K_1$ onto K_1). Write $C_X = \mathcal{D}_X + \tilde{X}$, where $\mathcal{D}_X = C_X \cap C_X^*$ and $\tilde{X} = (I - P_X)C_X P_X$ is isomorphic to X as a dual operator space. Similarly, we decompose $C_Y = \mathcal{D}_Y + \tilde{Y}$. Let $M \subseteq B(H_2 \oplus H_1, K_2 \oplus K_1)$ be a non-degenerate TRO such that $\overline{M C_X M^*}^{w^*} = C_Y$ and $\overline{M^* C_Y M}^{w^*} = C_X$. By [10, Proposition 2.8], we may also choose M to be a $\mathcal{D}_Y - \mathcal{D}_X$ -bimodule. Set $M_1 = P_Y M P_X \subseteq B(H_1, K_1)$ and $M_2 = (I - P_Y)M(I - P_X) \subseteq B(H_2, K_2)$. Since M is a $\mathcal{D}_Y - \mathcal{D}_X$ -module, we have that

$$M_1 M_1^* M_1 = P_Y M P_X M^* P_Y M P_X \subseteq P_Y (M M^* M) P_X \subseteq M_1,$$

and hence M_1 is a TRO. Similarly, we see that M_2 is a TRO.

Note that since $[M \mathcal{D}_X M^*]^* = [M \mathcal{D}_X M^*]$, we have that $M \mathcal{D}_X M^* \subseteq C_Y \cap C_Y^* = \mathcal{D}_Y$, and hence $M_1 A_X M_1^* \subseteq A_Y$ and $M_2 B_X M_2^* \subseteq B_Y$. Similarly, $M_1^* A_Y M_1 \subseteq A_X$ and $M_2^* B_Y M_2 \subseteq B_X$.

Finally, $P_Y [M \mathcal{D}_X M^*] (I - P_Y) = 0$, and it follows that

$$\begin{aligned}\tilde{Y} &= (I - P_Y)\mathcal{C}_Y P_Y = (I - P_Y)\overline{[M\mathcal{D}_X M^* + M\tilde{X}M^*]}^{w^*} P_Y \\ &= (I - P_Y)\overline{[M\tilde{X}M^*]}^{w^*} P_Y = \overline{[M_2\tilde{X}M_1^*]}^{w^*}.\end{aligned}$$

Similarly, $\tilde{X} = \overline{[M_2^*\tilde{Y}M_1]}^{w^*}$, and hence X and Y are TRO-equivalent. \square

Now we are ready to complete the proof of Theorem 2.5. Suppose that X and Y are dual operator spaces and that there exists a cardinal J such that $M_J(X) \cong M_J(Y)$ as dual operator spaces. We recall the unital dual operator algebras $\Omega(X)$ and $\Omega(Y)$ defined as in (2.1) and note that

$$M_J(\Omega(X)) \cong \begin{pmatrix} M_J(A_l(X)) & M_J(X) \\ 0 & M_J(A_r(X)) \end{pmatrix}.$$

By [6, Theorem 5.46(ii)], the algebras $M_J(A_l(X))$ and $A_l(M_J(X))$ are isomorphic as dual operator algebras, and it can be deduced from its proof that the isomorphism is given by sending a matrix $u = (u_{i,j}) \in M_J(A_l(X))$ to the multiplier T_u of $M_J(X)$ given by $T_u((x_{i,j})) = (\sum_k u_{i,k}(x_{k,j}))_{i,j}$. It follows from Proposition 2.1 that

$$\begin{pmatrix} M_J(A_l(X)) & M_J(X) \\ 0 & M_J(A_r(X)) \end{pmatrix} \cong \begin{pmatrix} A_l(M_J(X)) & M_J(X) \\ 0 & A_r(M_J(X)) \end{pmatrix}.$$

By Proposition 2.6, the algebra on the right-hand side is isomorphic as a dual operator algebra to

$$\begin{pmatrix} A_l(M_J(Y)) & M_J(Y) \\ 0 & A_r(M_J(Y)) \end{pmatrix}.$$

By the same arguments, this algebra is isomorphic to $M_J(\Omega(Y))$. It follows from Theorem 2.4 that the algebras $\Omega(X)$, $\Omega(Y)$ are Δ -equivalent as algebras. By Lemma 2.7(ii), X and Y are Δ -equivalent.

The proof of Theorem 2.5 is now complete. We note several immediate corollaries.

Corollary 2.8. *If A and B are unital dual operator algebras then the following are equivalent:*

- (i) A and B are Δ -equivalent as dual operator algebras;
- (ii) A and B are stably isomorphic as dual operator algebras;
- (iii) A and B are Δ -equivalent as dual operator spaces;
- (iv) A and B are stably isomorphic as dual operator spaces.

Proof. By Theorem 2.4, (i) is equivalent to (ii), while by Theorem 2.5, (iii) is equivalent to (iv). The equivalence of (ii) and (iv) follows from the generalized Banach–Stone theorem [3, Theorem 4.5.13]. \square

Since stable isomorphism is an equivalence relation we conclude:

Corollary 2.9. Δ -equivalence of dual operator spaces is an equivalence relation.

Let $A \subseteq B(H)$ and $B \subseteq B(K)$ be w^* -closed unital operator algebras and $M \subseteq B(H, K)$ be a TRO such that

$$A = \overline{M^*BM}^{w^*} \quad \text{and} \quad B = \overline{MAM^*}^{w^*}.$$

We define the $B - A$ -bimodule $X \stackrel{\text{def}}{=} \overline{MA}^{w^*} = \overline{BM}^{w^*}$ and the $A - B$ -bimodule $Y \stackrel{\text{def}}{=} \overline{AM^*}^{w^*} = \overline{M^*B}^{w^*}$. These bimodules are important in the theory of Δ -equivalence. In [11] they “generate” the functor of equivalence between the categories of normal representations of A and B . Also, it is proved in [13] that $B \simeq X \otimes_A^{\sigma_h} Y$ and $A \simeq Y \otimes_B^{\sigma_h} X$, where the tensor products are quotients of the corresponding normal Haagerup tensor products.

Corollary 2.10. *The spaces A, B, X, Y defined above are stably isomorphic.*

Proof. Observe that $M^*MAC \subseteq A$; hence

$$M^*XC \subseteq A \quad \text{and} \quad MAC \subseteq X.$$

It follows that X and A are TRO-equivalent. Similarly, we obtain that Y and B are TRO-equivalent. The claim now follows from Theorem 2.5. \square

In the special case of selfadjoint algebras we recapture the following known result:

Corollary 2.11. *Let A be a W^* -algebra and M be a w^* -closed TRO such that $A = \overline{M^*M}^{w^*}$. Then A and M are stably isomorphic.*

Proof. Observe that

$$MAC \subseteq \overline{MM^*M}^{w^*} \subseteq M \quad \text{and} \quad M^*MC \subseteq A.$$

It follows that A and M are TRO-equivalent. By Theorem 2.5, A and M are stably isomorphic. \square

In the next result we link the Δ -equivalence of two dual operator spaces X and Y to that of the corresponding algebras $\Omega(X)$ and $\Omega(Y)$.

Theorem 2.12. *The dual operator spaces X and Y are Δ -equivalent if and only if the algebras $\Omega(X)$ and $\Omega(Y)$ are Δ -equivalent.*

Proof. If X and Y are Δ -equivalent then there exists a cardinal I such that $M_I(X)$ and $M_I(Y)$ are isomorphic as dual operator spaces. Hence, $\Omega(M_I(X))$ and $\Omega(M_I(Y))$ are isomorphic as dual operator algebras. As in the proof of Theorem 2.5, using Proposition 2.1 and [6, Theorem 5.46(ii)], we conclude that

$$\Omega(M_I(X)) = \begin{pmatrix} A_l(M_I(X)) & M_I(X) \\ 0 & A_r(M_I(X)) \end{pmatrix} \cong \begin{pmatrix} M_I(A_l(X)) & M_I(X) \\ 0 & M_I(A_r(X)) \end{pmatrix} \cong M_I(\Omega(X))$$

and, similarly, $\Omega(M_l(Y)) \cong M_l(\Omega(Y))$. Thus, $\Omega(X)$ and $\Omega(Y)$ are stably isomorphic as algebras. By Theorem 2.4, $\Omega(X)$ and $\Omega(Y)$ are Δ -equivalent.

Conversely, if $\Omega(X)$ and $\Omega(Y)$ are Δ -equivalent then, by Lemma 2.7(ii), X and Y are Δ -equivalent. \square

Theorem 2.13. *Let X and Y be Δ -equivalent dual operator spaces. If (π, ϕ, σ) is a normal CES representation of the dual operator $A_l(X) - A_r(X)$ -module X and ϕ is a complete isometry, then there exists a normal completely isometric representation ψ of Y such that $\phi(X)$ is TRO-equivalent to $\psi(Y)$.*

Proof. The CES triple (π, ϕ, σ) defines a normal representation Φ of the algebra $\Omega(X)$. If $l \in A_l(X)$ with $\pi(l) = 0$ then $\phi(lx) = 0$ and hence $lx = 0$ for all $x \in X$. This implies that $l = 0$, and so π is one-to-one. Similarly σ is one-to-one. Thus, (π, ϕ, σ) is a faithful CES representation and induces the unique operator algebra structure on $\Omega(X)$. Thus, Φ is a normal completely isometric representation of the dual operator algebra $\Omega(X)$.

By Theorem 2.12, $\Omega(X)$ and $\Omega(Y)$ are Δ -equivalent; by [12, Theorem 2.7], there exists a normal completely isometric representation Ψ of $\Omega(Y)$ such that $\Phi(\Omega(X))$ is TRO-equivalent to $\Psi(\Omega(Y))$.

Let ψ be the restriction of Ψ to Y . By Lemma 2.7(i), the spaces $\phi(X)$ and $\psi(Y)$ are TRO-equivalent. \square

By [11], Δ -equivalence for dual operator algebras can be equivalently defined in terms of a special type of isomorphism between certain categories of representations of the algebras. These types of category isomorphisms are in the spirit of Morita equivalence. Thus, one would like to claim that the representations of $\Omega(X)$ and of $\Omega(Y)$ define certain special families of representations of X and Y such that X and Y are stably isomorphic if and only if these classes of representations are isomorphic. Unfortunately, the correspondence between representations of $\Omega(X)$ and representations of X is not one-to-one.

We finish this section with some applications of the above theorems.

Definition 2.3. An operator space X is called **rigid** if $M_l(X) = M_r(X) = \mathbb{C}$ and ***-rigid** if $A_l(X) = A_r(X) = \mathbb{C}$.

Note that if X is rigid, then it is *-rigid. There are many examples of rigid and *-rigid operator spaces. For example, the spaces $MAX(\ell_1^n)$ by a result of Zhang [21] (see also [16, Exercise 14.3]) can be identified with the subspace of the full group C^* -algebra of the free group on $n - 1$ generators, $C^*(\mathbb{F}_{n-1})$, spanned by the identity and the $n - 1$ generators. Moreover, Zhang argues that $I(MAX(\ell_1^n)) = I(C^*(\mathbb{F}_{n-1}))$ and since $C^*(\mathbb{F}_{n-1})$ is a C^* -subalgebra of its injective envelope it follows from [5, Theorem 1.9] that any left multiplier of $MAX(\ell_1^n)$ necessarily belongs to $I(C^*(\mathbb{F}_{n-1}))$ and multiplies the subspace $MAX(\ell_1^n)$ back into itself in the usual product. Since the identity belongs to the subspace, this forces the multiplier to be an element of the subspace and then it is easily seen that in fact it must be a multiple of the identity. A similar argument applies for right multipliers. Thus, $MAX(\ell_1^n)$ is rigid.

The argument given in the previous paragraph applies equally well to any subspace X of a unital C^* -algebra A which contains the identity and for which $I(X) = I(A)$. In this case, the left (and right) multipliers are simply the elements of the subspace X that leave the subspace

invariant under the algebra multiplication, and so it is often quite easy to determine whether X is rigid or $*$ -rigid.

Theorem 2.14. *Let X and Y be $*$ -rigid dual operator spaces. Then X and Y are stably isomorphic if and only if they are isomorphic as dual operator spaces.*

Proof. If X and Y are stably isomorphic, then they are Δ -equivalent. Hence, by Theorem 2.12, $\Omega(X)$ and $\Omega(Y)$ have completely isometric representations whose images are TRO-equivalent. The images of these representations are two concrete operator algebras \mathcal{C}_X and \mathcal{C}_Y of the type considered in Lemma 2.7, with A_X , B_X , A_Y and B_Y all scalar multiples of the identity and X and Y replaced by images of normal completely isometric representations, say $\phi(X)$ and $\psi(Y)$. Hence, the TRO's M_1 and M_2 appearing in the conclusions of Lemma 2.7(i), satisfy $M_1^* M_1 = M_1 M_1^* = M_2 M_2^* = M_2 M_2^* = \mathbb{C}$.

Now it readily follows that the spaces M_1 and M_2 are each the span of a single unitary. Let $M_i = \mathbb{C}U_i$, $i = 1, 2$, for some unitaries U_1 and U_2 . Applying Lemma 2.7 again, we see that $\psi(Y) = U_2^* \phi(X) U_1$ and the claim follows. \square

Corollary 2.15. *Let A and B be dual operator algebras for which $A \cap A^* = B \cap B^* = \mathbb{C}$. Then A and B are stably isomorphic as operator spaces if and only if they are isomorphic as dual operator algebras.*

Proof. Since B is a unital algebra, we have that $M_l(B) = M_r(B) = B$ and hence, $A_l(B) = A_r(B) = B \cap B^* = \mathbb{C}$. Hence, B , and similarly A , is a $*$ -rigid operator space. Thus, by Theorem 2.14, A and B are stably isomorphic if and only if they are isomorphic as dual operator spaces. By the generalized Banach–Stone theorem [3, Theorem 3.8.3], A and B are isomorphic as dual operator algebras. \square

It is interesting to note that the hypotheses and conclusions of the above corollary are really special to non-selfadjoint operator algebras. In fact, we now turn our attention to a special family of non-selfadjoint operator algebras to which our theory applies.

Definition 2.4. Let $G \subseteq \mathbb{C}^n$ be a bounded, connected, open set, i.e., a complex domain, and let $H^\infty(G) \subseteq L^\infty(G)$ denote the dual operator algebra of bounded analytic functions on G . We shall call G **holomorphically complete** if every weak*-continuous multiplicative linear functional on $H^\infty(G)$ is given by evaluation at some point in G .

Recall that two complex domains $G_i \subseteq \mathbb{C}^{n_i}$, $i = 1, 2$, are called **biholomorphically equivalent** if there exists a holomorphic homeomorphism, $\varphi: G_1 \rightarrow G_2$ whose inverse is also holomorphic.

Corollary 2.16. *Let G_i , $i = 1, 2$, be complex domains that are holomorphically complete. Then the following are equivalent:*

- (i) G_1 and G_2 are biholomorphically equivalent,
- (ii) $H^\infty(G_1)$ and $H^\infty(G_2)$ are isometrically weak*-isomorphic algebras,
- (iii) $H^\infty(G_1)$ and $H^\infty(G_2)$ are isometrically weak*-isomorphic dual Banach spaces,
- (iv) $H^\infty(G_1)$ and $H^\infty(G_2)$ are stably isomorphic dual operator spaces,

- (v) $H^\infty(G_1)$ and $H^\infty(G_2)$ are Δ -equivalent dual operator algebras,
- (vi) $H^\infty(G_1)$ and $H^\infty(G_2)$ are Δ -equivalent dual operator spaces.

Proof. Since G_1 and G_2 are connected sets, we have that $H^\infty(G_i) \cap H^\infty(G_i)^* = \mathbb{C}$, $i = 1, 2$. Also, since these algebras are subalgebras of commutative C^* -algebras, every contractive map between them is automatically completely contractive. Thus, the equivalence of (ii)–(vi) follows from the previous results.

Given a biholomorphic map $\varphi: G_1 \rightarrow G_2$, composition with ϕ defines the weak*-continuous isometric isomorphism between the algebras. Thus, (i) implies (ii).

Conversely, given a weak*-continuous isometric algebra isomorphism, $\pi: H^\infty(G_1) \rightarrow H^\infty(G_2)$, let $w \in G_2$, and let $E_w: H^\infty(G_2) \rightarrow \mathbb{C}$ denote the weak*-continuous, multiplicative linear functional given by evaluation at w . Then $E_w \circ \pi: H^\infty(G_1) \rightarrow \mathbb{C}$ is a weak*-continuous, multiplicative linear functional and hence is equal to E_z for some $z \in G_1$. If we assume that $G_1 \subseteq \mathbb{C}^n$, let z_1, \dots, z_n denote the coordinate functions on G_1 and set $\varphi_i = \pi(z_i) \in H^\infty(G_2)$, then it readily follows that $\varphi = (\varphi_1, \dots, \varphi_n): G_2 \rightarrow \mathbb{C}^n$ satisfies, $\varphi(w) = z$. Hence, $\varphi: G_2 \rightarrow G_1$. Since each of the mappings φ_i is holomorphic, a similar argument with the inverse of π shows that φ is a biholomorphic equivalence. Thus, (ii) implies (i). \square

Recalling that Δ -equivalence is originally defined in terms of a Morita-type equivalence of categories, we see that the equivalence of (i) and (v) shows that two domains have “equivalent” categories of representations in this sense if and only if they are biholomorphically equivalent. One does not need the full force of the rigidity result, Corollary 2.15, to prove the above result, since $H^\infty(G_i)$ is the center of $M_I(H^\infty(G_i))$ and any stable isomorphism must carry centers to centers. In fact, using the “isomorphism of centers” result of [14], one can replace (v) by the coarser Blecher–Kashyap equivalence.

3. Applications and examples

In this section we prove that whenever two dual operator algebras A and B are Δ -equivalent, there exists a dual operator space X such that A is completely isometrically isomorphic to $M_I(X)$ and B is completely isometrically isomorphic to $M_r(X)$. We then give an example of a dual operator space Y for which $M_I(Y)$ and $M_r(Y)$ are not stably isomorphic and hence not Δ -equivalent. We also give some examples which emphasize the difference between dual operator spaces arising from non-synthetic CSL algebras and those arising from synthetic ones.

Let $A \subseteq B(H)$ be a unital w^* -closed algebra, $\Delta(A) = A \cap A^*$ be its diagonal and $M \subseteq B(K, H)$ be a non-degenerate TRO such that $MM^* \subseteq A$. We call the space $X = \overline{AM}^{w^*}$ the **M -generated A -module**. In this section we fix A and M as above and we investigate some properties of X . Since $MM^* \subseteq A$ the space $B = \overline{M^*AM}^{w^*} \subseteq B(K)$ is a unital algebra and $XB \subseteq X$. Note that if we set $Y = \overline{M^*A}^{w^*}$, then A, X, Y, B form the four corners of what could potentially be a “linking” algebra of a Morita context. For this reason, we shall call (A, M, X, B) a **generating tuple**.

Theorem 3.1. *Let (A, M, X, B) be a generating tuple.*

- (i) $M_I(X)$ is isomorphic as a dual operator algebra to A and $M_r(X)$ is isomorphic as a dual operator algebra to B .

(ii) The algebra $\Omega(X)$ is isomorphic as a dual operator algebra to the algebra $D(X) = \begin{pmatrix} \Delta(A) & X \\ 0 & \Delta(B) \end{pmatrix}$.

Proof. Since $X \subseteq B(K, H)$, $A \subseteq B(H)$ and $AX \subseteq X$, by the definition of left multipliers, the map $\lambda(a) : X \rightarrow X$ given by $\lambda(a)(x) = ax$, is a left multiplier. It follows that the map

$$\lambda : A \rightarrow M_l(X) : a \rightarrow \lambda(a)$$

is contractive. It is also w^* -continuous by [3, Theorem 4.7.4].

We now prove that λ is an isometric surjection. Using analogous arguments, we can show that the map

$$\rho : B \rightarrow M_r(X), \quad \rho(b)(x) = xb$$

is w^* -continuous and contractive. Let u be in $M_l(X)$. By [3, Lemma 8.5.23] there exists a family $(m_i)_{i \in I} \subseteq M$ of partial isometries such that $m_i m_i^* \perp m_j m_j^*$ for $i \neq j$ and $I_H = \sum_{i \in I} m_i m_i^*$, the series converging in the strong operator topology. Let $x \in X$, $\xi \in K$ and $F \subseteq I$ be finite. Since the operators on X from $M_l(X)$ commute with those from $M_r(X)$ and since $M \subseteq X$ and $M^*X \subseteq B$, we have

$$\begin{aligned} \sum_{i \in F} u(m_i) m_i^* x(\xi) &= \sum_{i \in F} \rho(m_i^* x)(u(m_i))(\xi) = \sum_{i \in F} u(\rho(m_i^* x) m_i)(\xi) \\ &= \sum_{i \in F} u(m_i m_i^* x)(\xi) = u\left(\sum_{i \in F} m_i m_i^* x\right)(\xi). \end{aligned}$$

Since u is w^* -continuous [3, Theorem 4.7.1] we have that

$$\lim_F \sum_{i \in F} u(m_i) m_i^* x(\xi) = u(x)(\xi), \quad \xi \in K. \quad (3.1)$$

Observe that if $F = \{i_1, \dots, i_n\} \subseteq I$ then

$$\begin{aligned} \left\| \sum_{i \in F} u(m_i) m_i^* \right\| &= \left\| u((m_{i_1}, \dots, m_{i_n}))(m_{i_1}^*, \dots, m_{i_n}^*)^t \right\| \\ &\leq \|u\|_{M_l(X)} \|(m_{i_1}, \dots, m_{i_n})\| \|(m_{i_1}^*, \dots, m_{i_n}^*)^t\| \leq \|u\|_{M_l(X)}. \end{aligned}$$

Hence, the net $(\sum_{i \in F} u(m_i) m_i^*)_F$ is bounded. Since X is non-degenerate the limit of the net $(\sum_{i \in F} u(m_i) m_i^*(\xi))_F$ exists for all $\xi \in H$. We let $a = \sum_{i \in I} u(m_i) m_i^*$, the series converging in the strong operator topology. Since $XM^* \subseteq A$, we have that $a \in A$. Observe that

$$\|a\| \leq \|u\|_{M_l(X)}. \quad (3.2)$$

By (3.1), $ax = u(x)$ for all $x \in X$ and so $u = \lambda(a)$. We proved that λ is onto. By standard arguments, Eq. (3.2) implies that λ is isometric.

Let $n \in \mathbb{N}$ and $N = \underbrace{M \oplus \cdots \oplus M}_n$. Then the N -generated $M_n(A)$ -module is equal to $M_n(X) = \overline{[M_n(A)N]}^{w^*}$. By the arguments above, the map

$$\sigma : M_n(A) \rightarrow M_l(M_n(X)) : \sigma(a)(x) = ax$$

is a surjective isometry.

It follows from [6, Theorem 5.46(iii)] and its proof (see also [6, Eq. (2.4)]) that the mapping

$$L : M_n(M_l(X)) \rightarrow M_l(M_n(X)) : L((u_{ij})_{i,j})(x_{ij})_{i,j} = \left(\sum_k u_{ik}(x_{kj}) \right)_{i,j}$$

is a complete isometry. Since $\lambda^{(n)} = L^{-1} \circ \sigma : M_n(A) \rightarrow M_n(M_l(X))$, we have that λ is n -isometric. We have thus shown that λ is a completely isometry. Similarly, we can prove that ρ is completely isometric and surjective. By Proposition 2.1, the map

$$\Phi : D(X) \rightarrow \Omega(X) : \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \rightarrow \begin{pmatrix} \lambda(a) & x \\ 0 & \rho(b) \end{pmatrix}$$

is a dual operator algebra isomorphism. \square

Corollary 3.2. *If C and D are Δ -equivalent unital dual operator algebras then there exists a dual operator space X such that $C \cong M_l(X)$ and $D \cong M_r(X)$ as dual operator algebras.*

Proof. The algebras C and D have completely isometric normal representations which are TRO-equivalent. Letting A be the image of C , letting M be the TRO that induces the equivalence and applying Theorem 3.1 to the corresponding generating tuple completes the proof. \square

Remark 3.3. The converse of Corollary 3.2 does not hold. Example 3.9 shows that there exists a dual operator space Y such that $M_l(Y)$ and $M_r(Y)$ are not stably isomorphic.

Proposition 3.4. *Let (A, M, X, B) be a generating tuple. If Y is a dual operator space which is Δ -equivalent to the dual operator space X , then there exists a normal completely isometric representation ψ of Y such that X is TRO-equivalent to $\psi(Y)$.*

Proof. By Theorem 3.1, $\Delta(A)$ is isomorphic to $A_l(X)$, and $\Delta(B)$ is isomorphic to $A_r(X)$. Thus, there is a normal CES representation of the form (π, id_X, σ) of the dual operator $A_l(X) - A_r(X)$ -module X . Now apply Theorem 2.13. \square

We recall some definitions and concepts that we will need in the rest of the paper, see [8]. A **commutative subspace lattice (CSL)** is a strongly closed projection lattice \mathcal{L} whose elements mutually commute. A **CSL algebra** is the algebra $\text{Alg } \mathcal{L}$ of operators leaving invariant all projections belonging to a CSL \mathcal{L} . In the special case where \mathcal{L} is totally ordered we call \mathcal{L} a **nest** and the algebra $\text{Alg } \mathcal{L}$ a **nest algebra**. There exists a smallest w^* -closed algebra contained in \mathcal{A} which contains the diagonal $\Delta(\mathcal{A})$ of \mathcal{A} and whose reflexive hull is \mathcal{A} [1,19] (for the definition of the reflexive hull of an operator algebra see [8]). We denote this algebra by \mathcal{A}_{\min} . If $\mathcal{A} = \mathcal{A}_{\min}$ the CSL algebra is called \mathcal{A} **synthetic**.

Proposition 3.5. *Let A and D be CSL algebras and M and N be TRO's such that $MM^* \subseteq A$ and $NN^* \subseteq D$. Set $X = \overline{AM}^{w^*}$ and $Y = \overline{DN}^{w^*}$. Then X and Y are Δ -equivalent if and only if they are TRO-equivalent.*

Proof. Suppose that X and Y are Δ -equivalent. Since $\Omega(X)$ and $\Omega(Y)$ are Δ -equivalent, the algebras $D(X)$ and $D(Y)$ defined as in Theorem 3.1(ii) are Δ -equivalent. Assuming that $D(X)$ and $D(Y)$ are CSL algebras, it follows from [12, Theorem 3.2] that $D(X)$ and $D(Y)$ are TRO-equivalent. By Lemma 2.7(i), X and Y are TRO-equivalent.

We now prove that $D(X)$ is a CSL algebra, the proof for $D(Y)$ is similar. Denote by $pr(\mathcal{M})$ the set of all projections belonging to a von Neumann algebra \mathcal{M} . Let $B = [M^*AM]^{-w^*}$. By [10, Proposition 2.8], we may assume that $\Delta(B) = [M^*M]^{-w^*}$, $\Delta(A) = [MM^*]^{-w^*}$. Since $\Delta(A)$ contains a maximal abelian selfadjoint algebra (MASA), we can easily check that $\Delta(B)$ also contains a MASA and so the algebra $D(X)$ contains a MASA. It now suffices to prove that $D(X)$ is a reflexive space. Suppose that $A \subseteq B(H)$, $B \subseteq B(K)$. Since the invariant subspace lattice $\text{Lat}(D(X))$ of $D(X)$ is contained in $\Delta(A)' \oplus \Delta(B)'$, we can verify that the reflexive hull $\text{Ref}(D(X))$ of $D(X)$ is the space of $w \in B(H \oplus K)$ satisfying

$$\forall e_i \in pr(\Delta(A)'), f_i \in pr(\Delta(B)'), \\ (e_1 \oplus f_1)D(X)(e_2 \oplus f_2) = \{0\} \Rightarrow (e_1 \oplus f_1)w(e_2 \oplus f_2) = 0.$$

If $w = \begin{pmatrix} u & a \\ 0 & v \end{pmatrix} \in \text{Ref}(D(X))$ and $e_1, e_2 \in pr(\Delta(A)'),$ such that $e_2\Delta(A)e_1 = 0$, then

$$(e_2 \oplus 0)D(X)(e_1 \oplus 0) = \{0\} \Rightarrow (e_2 \oplus 0)w(e_1 \oplus 0) = 0 \Rightarrow e_2ue_1 = 0.$$

Hence, $u \in \Delta(A)$. Similarly, $v \in \Delta(B)$. If $e \in pr(\Delta(A)'), f \in pr(\Delta(B)'),$ satisfy $eXf = \{0\}$, then

$$(e \oplus 0)D(X)(0 \oplus f) = \{0\} \Rightarrow eaf = 0.$$

Thus, it follows that $a \in \text{Ref}(X) = X$. We have shown that $w \in D(X)$, and hence $\text{Ref}(D(X)) = D(X)$. Thus, $D(X)$ is reflexive and contains a MASA, and hence, $D(X)$ is a CSL algebra. \square

Example 3.6. We now give an example of spaces which are not Δ -equivalent. Let A be a CSL algebra, B be a non-synthetic, separably acting CSL algebra and M and N be TRO's such that $MM^* \subseteq A$ and $NN^* \subseteq B$. Then the spaces $X = \overline{AM}^{w^*}$ and $Y = \overline{B_{\min}N}^{w^*}$ are not Δ -equivalent. Indeed, if they were, they would be stably isomorphic. On the other hand, Corollary 2.10 implies that X is stably isomorphic to A and Y is stably isomorphic to B_{\min} . Thus, the algebras A and B_{\min} would be stably isomorphic, hence Δ -equivalent. This contradicts [12, Theorem 3.4].

Let \mathcal{N}_1 and \mathcal{N}_2 be nests acting on separable Hilbert spaces H_1 and H_2 , respectively. Recall [8] that \mathcal{N}_1 and \mathcal{N}_2 are called similar if there exists an invertible operator $y: H_1 \rightarrow H_2$ such that $\mathcal{N}_2 = \{yn(H_1): n \in \mathcal{N}_1\}$. In this case there exists an order isomorphism $\theta: \mathcal{N}_1 \rightarrow \mathcal{N}_2$ which preserves the dimension of the atoms of \mathcal{N}_1 and \mathcal{N}_2 , namely, $\theta(n)$ can be taken to be equal to the projection onto $yn(H_1)$, for all $n \in \mathcal{N}_1$. We say that the invertible operator $y \in B(H_1, H_2)$ implements θ . Let

$$Y = \{y \in B(H_1, H_2): (I - \theta(n))yn = 0, \forall n \in \mathcal{N}_1\}$$

and

$$Z = \{x \in B(H_2, H_1) : (I - n)x\theta(n) = 0, \forall n \in \mathcal{N}_1\}.$$

If $C = \text{Alg } \mathcal{N}_1$ and $D = \text{Alg } \mathcal{N}_2$ one can easily verify that

$$C = [\overline{ZY}]^{w*}, \quad D = [\overline{YZ}]^{w*}, \quad CZD \subseteq Z \quad \text{and} \quad DYC \subseteq Y.$$

We will need the Similarity Theorem [8, Theorem 13.20]:

Theorem 3.7. *For every $\delta > 0$ there exists an invertible operator $y \in Y$ which implements θ such that $\|y\| < 1 + \delta$ and $\|y^{-1}\| < 1 + \delta$.*

Theorem 3.8.

- (i) $M_l(Z) \cong C$, $M_r(Z) \cong D$ as dual operator algebras.
- (ii) The algebra $\Omega(Z)$ is isomorphic as a dual operator algebra to the algebra $\begin{pmatrix} \Delta(C) & Z \\ 0 & \Delta(D) \end{pmatrix}$.

Proof. We can easily check that the map

$$\tau : C_2(Z) \rightarrow C_2(Z) : (x_1, x_2)^t \rightarrow (ax_1, x_2)^t$$

is completely contractive for all $a \in C$. So by [3, Theorem 4.5.2] the linear map $\lambda : C \rightarrow M_l(Z)$ given by $\lambda(a)(x) = ax$ is contractive. Moreover, λ is one-to-one. To see this, suppose that $\lambda(a) = 0$ for some $a \in C$. Then $ax = 0$ for all $x \in Z$, and hence $axy = 0$ for all $x \in Z$ and all $y \in Y$. Since $C = [\overline{ZY}]^{w*}$, this implies that $a = 0$. Similarly, the map $\rho : D \rightarrow M_r(Z)$ given by $\rho(b)(x) = xb$ is contractive. The maps λ and ρ are w^* -continuous by [3, Theorem 4.7.4].

Let u be in $M_l(Z)$. By Theorem 3.7 for every $\delta > 0$ there exists $y_\delta \in Y$ such that $y_\delta^{-1} \in Z$, $\|y_\delta\| < 1 + \delta$ and $\|y_\delta^{-1}\| < 1 + \delta$. Since the operators of $M_l(Z)$ and $M_r(Z)$ commute, for all $x \in X$ we have

$$u(y_\delta^{-1})y_\delta x = \rho(y_\delta x)(u(y_\delta^{-1})) = u(\rho(y_\delta x)y_\delta^{-1}) = u(y_\delta^{-1}y_\delta x) = u(x).$$

If $a_\delta = u(y_\delta^{-1})y_\delta \in C$ then for all $\delta > 0$ we have that $\lambda(a_\delta) = u$. It follows that λ is surjective. Since λ is one-to-one, $a_\delta = a$, for all δ and

$$\|a\| = \|u(y_\delta^{-1})y_\delta\| \leq \|u\|_{M_l(Z)}(1 + \delta)^2, \quad \text{for all } \delta > 0,$$

we have that $\|a\| \leq \|u\|_{M_l(Z)}$. Thus, λ is isometric.

If $n \in \mathbb{N}$ the algebras $M_n(C)$, $M_n(D)$ are similar nest algebras. Repeating the above arguments we can show that λ is n -isometric. Hence λ , and similarly ρ , are completely isometric. \square

Example 3.9. The above result shows that there exists a dual operator space Z such that $M_l(Z)$ and $M_r(Z)$ not stably isomorphic. Indeed, from [12, Example 3.7] there exist similar nest algebras C and D which are not stably isomorphic. The claim now follows from Theorem 3.8.

References

- [1] W.B. Arveson, Operator algebras and invariant subspaces, *Ann. of Math.* (2) 100 (1974) 433–532.
- [2] D.P. Blecher, U. Kashyap, Morita equivalence of dual operator algebras, *J. Pure Appl. Algebra*, in press.
- [3] D.P. Blecher, C. Le Merdy, *Operator Algebras and Their Modules—An Operator Space Approach*, Oxford University Press, 2004.
- [4] D.P. Blecher, P.S. Muhly, V.I. Paulsen, Categories of operator modules (Morita equivalence and projective modules), *Mem. Amer. Math. Soc.* 143 (2000), no. 681.
- [5] D.P. Blecher, V.I. Paulsen, Multipliers of operator spaces, and the injective envelope, *Pacific J. Math.* 200 (2001) 1–17.
- [6] D.P. Blecher, V. Zarikian, The calculus of one-sided M -ideals and multipliers in operator spaces, *Mem. Amer. Math. Soc.* 179 (2006), no. 842.
- [7] L. Brown, P. Green, M. Rieffel, Stable isomorphism and strong Morita equivalence of C^* -algebras, *Pacific J. Math.* 71 (1977) 349–363.
- [8] K.R. Davidson, *Nest Algebras*, Longman Scientific & Technical, Harlow, 1988.
- [9] E. Effros, Z.-J. Ruan, *Operator Spaces*, Clarendon Press, Oxford, 2000.
- [10] G.K. Eleftherakis, TRO equivalent algebras, preprint, arXiv:math.OA/0607488.
- [11] G.K. Eleftherakis, A Morita type equivalence for dual operator algebras, *J. Pure Appl. Algebra* 212 (5) (2008) 1060–1071.
- [12] G.K. Eleftherakis, Morita type equivalences and reflexive algebras, *J. Operator Theory*, in press.
- [13] G.K. Eleftherakis, V.I. Paulsen, Stably isomorphic dual operator algebras, *Math. Ann.* 341 (1) (2008) 99–112.
- [14] U. Kashyap, A Morita theorem for dual operator algebras, *J. Funct. Anal.* 256 (11) (2009) 3545–3567.
- [15] K. Morita, Duality of modules and its applications to the theory of rings with minimum condition, *Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A* 6 (1958) 85–142.
- [16] V.I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, 2002.
- [17] G. Pisier, *Introduction to Operator Space Theory*, Cambridge University Press, 2003.
- [18] M. Rieffel, Morita equivalence for C^* -algebras and W^* -algebras, *J. Pure Appl. Algebra* 5 (1974) 51–96.
- [19] V.S. Shulman, L. Turowska, Operator synthesis. I. Synthetic sets, bilattices, tensor algebras, *J. Funct. Anal.* 209 (2004) 293–331.
- [20] C.-Y. Suen, Completely bounded maps on C^* -algebras, *Proc. Amer. Math. Soc.* 93 (1985) 81–87.
- [21] C. Zhang, Representations of operator spaces, *J. Operator Theory* 33 (1995) 327–351.